

CHILDREN'S LINGUISTIC AND NUMERIC UNIT COORDINATION

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Detailing is a linguistic tool for mathematical argumentation in which given mathematical information is operationalized through one's warrants to support a claim. Recent literature suggests that students' detailing is related to their early algebraization. This study examined 168 elementary students' use of detailing in two mathematical argumentative tasks in relation to their enacted scheme of multiplicative unit coordination. A convergent mixed methods approach was used to analyze students' argumentative writing qualitatively, and merge these findings with quantitative indicators for students' multiplicative reasoning. Results suggest a statistically significant relationship between students' detailing and their multiplicative unit coordination.

Keywords: Elementary School Education, Number Concepts and Operations, Reasoning and Proof

The development of mathematical argument in the elementary grades is relatively undertheorized. Although various descriptions and models of how children come to engage in more sophisticated mathematical argumentation and/or proof have been proposed (Blanton & Kaput, 2011; Krummheuer, 2007; Morris, 2009; Tall et al., 2011), the vast majority of such descriptions focus only on generalized arguments. This bias towards generalization is understandable, given the necessity of generalization in the desired development of proof processes. However, it has neglected aspects of children's mathematical argumentation that may precede inference and generalization. Following Peirce, Kosko (2016) defined the purpose of mathematical argument as establishing an acceptable mathematical claim of truth. According to Peirce (1903/1998), argument necessarily involves inference for generalization. Yet, the first step of argumentation, colligation, precedes inference. *Colligation* occurs when an individual conveys the collective warrants for their argument as a singular copulative proposition. The copulative proposition can then be used to support a generalized claim. For example, in proving *the sum of two consecutive odd integers is divisible by two*, an individual may use the equation $n + (n + 2) = 2n + 2 = 2(n + 1)$ as part of their proof. The equation includes several propositions that, collectively, support the claim to be proven. Colligation in mathematical argument is facilitated by various linguistic tools (Kosko & Singh, 2016a). However, this particular study focuses specifically on the linguistic tool of detailing, in which the given information from a task is operationalized via a reference chain as a means of providing cohesion for a copulative proposition (Kosko, 2016; Kosko & Singh, 2016a).

Recent study of children's detailing in mathematical argument has identified a potential relationship with abstraction of number (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Singh, 2016b). Specifically, children's abstraction in unit coordination of number has been found to relate to their abstraction of linguistic information units in argumentation. In the present study, we seek to investigate this phenomenon further by studying children's engagement in detailing in relation to their demonstrated unit coordination in multiplicative contexts. Therefore, the purpose of this study is to examine children's detailing enacted in MAW across several tasks in relation to their enacted multiplicative coordination of units.

Theoretical Framework

Detailing as Colligation in Children's Mathematical Argumentative Writing

The present study takes a Peircian semiotic view of mathematical argument as a theoretical lens, while applying Systemic Functional Linguistics (SFL) as an analytic lens. Peirce (1903/1998)

defined *argument* as a sign that synthesizes various propositions to establish a generalizable claim. Further, arguments synthesize other abstracted signs including, but not limited to, copulative signs. *Copulative signs* are the synthesized set of propositions used to foreground the inference towards a generalizable claim. As such, copulative signs are part of arguments, but do not include logical inference or a generalizable claim. The formation of a copulative sign involves the action of colligation, or the collective expression of propositions into a singular copulative proposition. Colligation is facilitated by various linguistic tools. Kosko and colleagues (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Singh, 2016b; Kosko & Zimmerman, 2015) have identified at least two such linguistic tools: nominalization and detailing. Nominalization occurs when two or more mathematical linguistic objects are metaphorically conveyed as one (Halliday & Matthiessen, 2004). For example, $4n+2$ is considered as a singular expression, but includes the discrete nominal objects $4n$ and 2 as being summed. Further, $4n$ could be considered as the product of two discrete nominal objects (4 and n). Thus, nominalization facilitates colligation as a tool for abstraction of multiple nominal objects. Although nominalization is useful, and often essential, for colligation, the present study limits its focus to the linguistic tool of detailing.

Kosko (2016) describes detailing as the operationalization of given information through the construction of a reference chain through the warranted propositions supporting a claim. According to Halliday and Matthiessen's (2004) approach to SFL, reference serves the role of establishing cohesion for a text. In this manner, reference chains can be used to establish new information and re-establish given information for conveying and establishing *information units*. Information units are grammatical units that establish new information based on given information. Detailing goes beyond more common applications of reference chains in that given information is continuously operationalized and built up over multiple warranted propositions (i.e., information units). The following third grade student's mathematical argumentative writing provides an example of detailing. The child was asked to respond to the Cuisenaire-rods based task *If a red rod is 5, a yellow rod can't be 9 because...* Elements of the detailed reference chain are in bold, with grammatical clauses separated by "//":

The yellow rod can't be 9 // because **1 red one is 5** // and **that** does not take it all up // so you put **another 5** // and **it's** still not big enough // but **it [is] basically 10** // and **it's** smaller than the yellow // so the yellow can't be 9.

The child's writing begins and ends with their non-generalized claim that a yellow can't be 9 long. The clauses between serve the role of the copulative proposition, which is cohesively bound by a detailed reference chain. The given, **1 red one is 5**, is operationalized in the information unit **it [is] basically 10**. The operationalized referent, **10**, is exophoric in that it presents new information not in the information unit. Although technically new information, **10** is established in the information unit from the given information in a manner that represents a transformed, or operationalized, version of the given referent. Thus, **10** is part of a detailed reference chain that is used to establish the claim of yellow not being 9 long. Halliday and Matthiessen's (2004) definition identifies both referents as different information units, but detailing allows for the formation of a unique type of reference chain that establishes a copulative proposition. Specifically, **it [is] basically 10** serves as a link in the detailed reference chain connected to the given **1 red one is 5** as well as the referent in the proposition "**it's** smaller than the yellow." Although each proposition provides a different exchange of given and new information, each new information referent is a transformed version of the given information allowing for cohesion. Thus, detailing allows for all three propositions to collectively serve as a copulative proposition to support the child's claim.

As can be gathered from the preceding description, detailing allows for the linguistic coordination and cohesion of different information units into a singular copulative proposition. The primary aim of the present study is to investigate whether children's choice of detailing (i.e.,

linguistic unit coordination) coincides with their demonstrated ability to coordinate mathematical units (i.e., number). Thus, we briefly discuss multiplicative unit coordination to foreground a description of a conjectured relationship between these two concepts.


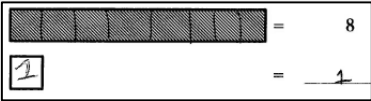
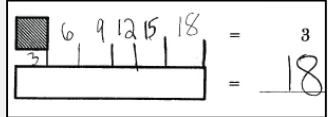

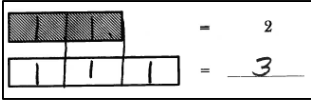
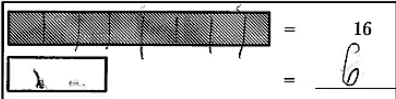
Children's Multiplicative Unit Coordination

This study considers multiplicative reasoning from the perspective of scheme theory, with specific focus on Hackenberg's (2010) multiplicative concepts. According to Steffe (1994) multiplicative schemes require the coordination of at least two levels of units, and to develop multiplicative concepts students require the anticipatory use of such schemes. Specifically, students may construct schemes, or collections of actions, *in activity* (as they are engaged in a task), or they may use them in anticipation of the actions they expect to engage with the task (i.e., *anticipatory schemes*). Hackenberg (2010) suggests three stages of students' multiplicative concepts with the subsequent concept requiring more the use of anticipatory schemes where in activity schemes may have been used previously. The transition from less to more sophisticated multiplicative concepts is gradual, and intermediate levels are developed successively through internalizing in-activity schemes such that they begin to be used as anticipatory schemes (Norton et al., 2015).

Anticipatory and in-activity schemes described above are based on qualitative assessments and assume that a student transitions gradually from enacting schemes in activity, to using schemes in an anticipatory manner. However, examination of whether students enact multiplicative schemes either in activity or in an anticipatory manner incorporates a depth of analysis that requires more time than can be devoted when considering larger sample sizes. Such is the case in the present study ($n = 168$). Therefore, we adopt the approach of Kosko and Singh (in review), in which evidence of schemes *enacted* by students is identified from written work. Such evidence is, by its nature, an artifact of the activity generally observed by qualitative teaching experiments. Therefore, enacted schemes, as defined in the present study, do not distinguish between in-activity and anticipatory schemes. However, enacted schemes are further assessed with enacted reversible schemes to attempt a closer representation of anticipatory schemes than in-activity. Although enacted schemes do not allow for the critical distinction between in-activity and anticipatory schemes, their use does allow for larger scale data collection (Kosko & Singh, in review).

Following Hackenberg's (2010) three multiplicative concepts, but with the caveat of using enacted schemes in place of in-activity and anticipatory, the present study considers students' multiplicative reasoning via four tiers. The tiers align with Hackenberg's (2010) multiplicative concepts. Multiplicative Tier 0 (MT-0) involves pre-multiplicative schemes (i.e., counting by 1s with records of counting). The other three tiers (MT-1, MT-2, and MT-3) are similar to Hackenberg's three stages respectively with the primary difference that the basis of classification in our case is the enacted rather than the anticipatory scheme. In other words a student belonging to MT-1 can be considered to fall in Hackenberg's first multiplicative concept. Thus the first multiplicative tier (MT-1) involves the enacted coordination of two levels of units, and MT-2 involves the enacted coordination of three levels of units. MT-3 involves the enacted coordination of three levels of units, as does MT-2, but shows clear evidence of disembedding with non-1 units, and is considered more likely to correspond to Hackenberg's (2010) definition of the third multiplicative concept.

Table 1: Students' Unit Coordination and Enacted Schemes

Tiers	Students' ways of Unit coordination	Enacted schemes	Examples
MT-0	No unitization or unit coordination	Iterating 1 units n times	
MT-1	Students' may coordinate two levels of units	Partitioning into n parts to find 1 units. Iterating non-1 units n times	 
MT-2	Students' may coordinate three levels of units	Partitioning into n parts to find non-1 units. Disembedding a unit to iterate n times.	 
MT-3	Students' can coordinate with three levels of units even with rational numbers.	Decompose partitions into non-1 units (may include coordination of partitions in both length models).	

Coordination of Mathematical Quantitative and Mathematic-Linguistic Information Units

Recent examinations of elementary children's use of detailing have found relationships between whether students engage in detailing and their success on tasks requiring multiplicative reasoning (Kosko, 2016; Kosko & Singh, 2016a), as well as the emergence of nominalization in mathematical argument and the development of relational conceptions of equivalence (Kosko & Singh, 2016b). Such findings follow prior research identifying relationships between students' generalizations and their success with early algebra tasks (e.g., Blanton & Kaput, 2011; Morris, 2009), but examines the interplay between the development of mathematical argumentation and early algebra at a finer gran size of analysis. The present study posits an even more specified view of this interplay. Specifically, children's use of detailing colligates information units to create copulative signs (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Zimmerman, 2015). Children's creation of copulative signs points to an ability to abstract multiple meanings as one. In a very similar fashion, children's coordination of quantitative units in ways mathematics education researchers would consider as multiplicative also points to an ability to abstract multiple meanings as one. Although these two types of coordination point to the same ability to abstract meaning, we conjecture that they are not necessarily the same, but similar enough to co-occur more often than not. However, such a potential interplay is important, given the potential of one type of coordination to influence the development of the other (Kosko, 2016).

The interplay conjectured here provides a much needed mechanism for explaining observed co-development of early algebra and mathematical argument, with a specific focus on multiplicative reasoning and detailing. Because detailing involves the coordination of multiple information units into what Pierce (1903/1998) refers to as a copulative proposition, and multiplicative reasoning at its initial tiers involves the coordination of multiple non-1 quantities into a singular number, we conjecture that both forms of coordination involve a similar form of abstraction. Thus, they should be more likely to co-occur than not. Our rationale for this conjecture lay in the nature of both kinds of coordination. Specifically, information units are composed of multiple nominal elements such that each proposition must be considered, at least tacitly, as its own entity. Similarly, multiplicative reasoning requires that a non-1 unit be considered at least as a grouping of 1s that can be operated upon. Coordination of propositions (in the form of information units) via detailing requires a level of linguistic coordination beyond simply providing a sequence of propositions. Rather, the propositions must be cohesively joined, and this is an aspect of linguistic coordination that seems to become more prevalent beginning in the second grade (Kosko & Zimmerman, 2015). Likewise, multiplicative reasoning involves the coordination of number beyond counting by 1s; a form of unit coordination that appears to begin emerging more prevalently in second and third grade (Kosko & Singh, in review). Therefore, the level of coordination involved in detailing and multiplication is similar in that both move beyond the basic operations of their domains, but involve at least some movement to coordination units of units. Should our conjecture hold true, we would anticipate seeing a relationship between children's multiplicative reasoning and their detailing across different types of tasks. Thus, we ask the following research question: How does children's multiplicative unit coordination relate to the presence of detailing in their mathematical argumentative writing?

Methods

Sample and Measures

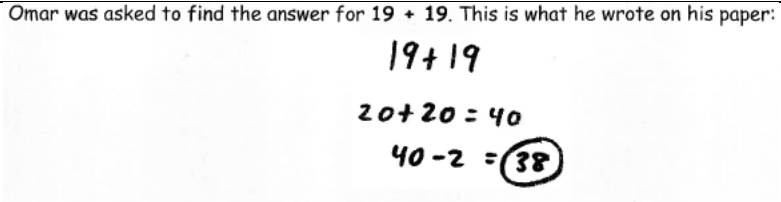
Data were collected in May 2015 from 168 second and third grade students in two suburban school districts in a Midwestern U.S. state. Second grade students were enrolled in four different teachers' classrooms ($n = 76$) and third grade students were enrolled in three different teachers' classrooms ($n = 92$). Participants completed two packets across two weeks. In week one, the packet included a multiplicative reasoning assessment. In week two, the packet included six mathematical argumentative writing tasks, although we limit our discussion in the present paper to only two tasks for sake of space and simplicity (Tasks 3 and 4).

The multiplicative reasoning assessment was developed by Kosko and Singh (in review) and includes 12 items designed to assess students' use of different enacted unit coordination schemes. We used a rubric involving 11 codes to assess students' demonstrated work (Kosko & Singh, in review). The students were assigned to a particular tier if they received correct scores to over half of the items in that tier. As an example we assigned students to MT-2 who got 3 out of 4 items (aligned with MT-2) correct, provided they also meet the criteria for lower tiers. Our analysis placed 67.5% students in MT-0, 18.4 % in MT-1, and 14.1 % in MT-2. None of the students were placed in MT-3. A Cronbach's alpha coefficient of .86 was calculated for the enacted schemes in all 12 items across different tiers, suggesting sufficient reliability of the assessment (for further information on the assessment, see Kosko & Singh, in review).

The mathematical argumentative writing tasks included six tasks with three incorporating length model representations of arithmetic and three incorporating the use of expressions and equations for arithmetic. The two tasks discussed in the present paper are presented in Table 2. Task 3 required students to use Cuisenaire rods (a color-coded length model) and assume that a red rod is 5 long, although it cannot be physically partitioned into 5 sub-units. Kosko (2016) found that use of this task encouraged more detailing than a similar task in which allowed for such physical partitioning.

Similarly, Task 4 requires students to consider three separate expressions/equations in relation to one another. To do so, it was assumed that detailing would be necessarily enacted to communicate the strategy cohesively (i.e., that each expression/equation serves as an information unit that can be colligated via detailing).

Table 2: Mathematical Argumentative Writing Tasks

Task 3*	Task 4
<p><i>If a red rod is 5 long, a yellow rod can't be 9 because...</i></p>	<p>Omar was asked to find the answer for $19 + 19$. This is what he wrote on his paper:</p>  <p>Explain how Omar solved the problem and why it does (or doesn't) work.</p>

*Students completed task 3, and other length model tasks, with Cuisenaire rods.

Analysis

The present study incorporates a data-transformation variant of convergent mixed methods design (Creswell & Plano Clark, 2011). Specifically, an SFL approach to examining functional grammar was used to qualitatively analyze second and third grade students' use of reference and detailed reference chains in their mathematical argumentative writing. Findings were organized into classifications that were quantified into variables for Chi-Square analysis. Quantitative data was collected from the multiplicative reasoning assessment and merged with the quantified SFL analysis for the mixing of data. Un-quantified qualitative findings were then used to help interpret quantitative findings from the Chi-Square analysis.

Qualitative analysis of functional grammar. We used SFL to examine the presence and patterns of reference use in children's mathematical argumentative writing. Reference is part of the textual metafunction of grammar, with its primary role to promote cohesion and coherence to an audience (Halliday & Matthiessen, 2004). A child communicating mathematically may use isolated referents, or may use reference chains that link referents in two or more propositions. In the present study, Kosko's (2016) description of detailing is used to distinguish between general reference chains and detailed reference chains. Detailed reference chains were coded for the operationalization of initial given referents in a manner that colligated two or more information units. Information units are clause-level grammatical units that connect given and new referents to convey information (Halliday & Matthiessen, 2004). Detailing, as defined in this paper, creates reference chains with endophoric and exophoric references that allow for two or more information units to be considered holistically as a colligated proposition.

The structure of mathematical writing tasks typically provides given information and an indication for a claim to be established; similar to Herbst and Chazan's (2011) described norms for providing a proof problem in high school Geometry. This structure allows for the construction of hypothetical reference chains that are more likely to be written to form a copulative proposition. For Task 3 (see Table 2), a complete and detailed reference chain should include a reference to the red rod being 5 long, operationalization of this given stating that two reds are 10 long, and an extension of this latter proposition to convey that 10 cannot be less than 9. Presence of variations of these three propositions was coded as detailing, but no other variations of detailing were observed. Both authors examined data for the initial information unit, incomplete detailing (i.e., two but not all three information units present), and complete detailing. Data were quantified, and coding was found to have strong interrater reliability ($K=.67$). Coding was then dichotomized to compare prevalence of detailing and not detailing in the quantitative analysis ($M=.22$, $SD=.42$).

Similar to coding of Task 3, Task 4 had one identified and coded detailed reference chain including two information units: a proposition conveying that Omar added 1 to each 19, and a proposition conveying that 2 be subtracted from the total. Incomplete detailing was observed when referents were not operationalized in a manner to colligate the two information units. As with the prior discussed task, data were quantified and coding was found to have strong interrater reliability ($K=.90$). Data were then dichotomized to compare prevalence of detailing and not detailing ($M=.38$, $SD=.49$).

Quantitative analysis of multiplicative reasoning and detailing. Chi-Square statistics were calculated to investigate whether the presence of detailing in students' MAW coincided with their multiplicative tiers. The relationship between detailing and multiplicative tier was found to be independent from chance both for Task 3 ($\chi^2(df=2) = 8.121, p = 0.017$) and Task 4 ($\chi^2(df=2) = 18.971, p = 0.000$). In order to better understand these findings, a post hoc analysis was used to examine the differences between specific observed and expected frequencies within individual cells of the Chi-Square contingency table. The adjusted standardized residuals for each cell in the 2x3 table (i.e., dichotomous detailing code X three observed multiplicative tiers) were calculated, and represent a statistic similar to a z-score for the difference between observed and expected counts in each cell (Haberman, 1973). Critical values for adjusted standardized residuals were ± 2.0 . Statistically significant and positive residuals were observed for the presence of detailing for students at MT-2 for Task 3 (2.5) and Task 4 (3.1), as well as for students at MT-1 for Task 4 (2.4). Students at MT-0 were observed to engage in detailing less than expected by chance for both Task 3 (-2.6) and Task 4 (-4.3). Therefore, the multiplicative tier a student was placed was found to coincide with the presence or absence of detailing, with the observed frequencies are outside those expected by chance. However, differences in magnitude of adjusted standardized residuals suggest that the relationship may vary by task.

Discussion and Conclusion

Findings from this paper suggest that elementary children's enactment of detailing in their mathematical argumentative writing on two tasks is not independent from their demonstrated ability to coordinate units multiplicatively. However, it is important to note that both analyzed tasks solicited detailing that directly linked endophoric with exophoric references between propositions. It is likely that some tasks may solicit less (or more) sophisticated reference chains. For example, some mathematical writing tasks may solicit reference chains that link only endophoric references together, which is less linguistically complex than detailing solicited from the tasks in this paper. It is also feasible that a task may solicit the linking of exophoric references between two information units, providing a more linguistically complex example of detailing than presented in this study. Such variations of detailing and reference use in mathematical argument are in need of further study. Although the present study did not explore such variations, there are specific and significant implications of these findings.

Earlier in this paper, we conjectured that linguistic coordination of information units via detailing and unit coordination via demonstrated multiplicative reasoning were similar enough in their observable structure to co-occur among elementary children. The findings of the present study provide additional evidence for this conjecture, as does recent work in this area (Kosko, 2016; Kosko & Singh, 2016b). Although these two types of coordination point to the same ability to abstract meaning, we conjecture that they are not necessarily the same. From a theoretical perspective, this suggests we should not expect a students' demonstrated ability with one coordination type to automatically coincide with the other. Yet from a practical perspective, findings here suggest that the similarity of each coordination type may allow for improved instruction of each via incorporation of the other.

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